Bisolutions to the Klein–Gordon Equation and Quantum Field Theory on 2-dimensional Cylinder Spacetimes

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Abstract

We consider 2-dimensional cylinder spacetimes whose metrics differ from the flat Minkowskian metric within a compact region K. By choice of time orientation, these spacetimes may be regarded as either globally hyperbolic *timelike cylinders* or nonglobally hyperbolic *spacelike cylinders*. For generic metrics in our class, we classify all possible candidate quantum field algebras for massive Klein–Gordon theory which obey the F-locality condition introduced by Kay. This condition requires each point of spacetime to have an intrinsically globally hyperbolic neighbourhood N such that the commutator (in the candidate algebra) of fields smeared with test functions supported in N agrees with the value obtained in the usual construction of Klein–Gordon theory on N.

By considering bisolutions to the Klein–Gordon equation, we prove that generic timelike cylinders admit a unique F-local algebra – namely the algebra obtained by the usual construction – and that generic spacelike cylinders do not admit any F-local algebras, and are therefore non F-quantum compatible. Refined versions of our results are obtained for subclasses of metrics invariant under a symmetry group. Thus F-local field theory on 2-dimensional cylinder spacetimes essentially coincides with the usual globally hyperbolic theory. In particular the result of the author and Higuchi that the Minkowskian spacelike cylinder admits infinitely many F-local algebras is now seen to represent an anomalous case.

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1 Introduction

During the last few years there has been much interest in classical and quantum dynamics on nonglobally hyperbolic spacetimes, with attention focussing on those spacetimes which contain closed timelike curves and which therefore provide mathematical models for the concept of a 'time machine'. The usual formulation of quantum field theory breaks down on these spacetimes owing to the absence of global advanced-minus-retarded fundamental solutions to the corresponding classical field equations, so any study of quantum theory on nonglobally hyperbolic spacetimes must be undertaken within a generalisation of the usual theory. In this paper, we will study quantum field theory on both globally hyperbolic and nonglobally hyperbolic 2-dimensional cylinder manifolds within a framework for quantum field theory on not-necessarily globally hyperbolic spacetimes proposed by Kay [1]. (See [2, 3, 4, 5] and references therein for other approaches to the formalism and interpretation of quantum theory in the presence of chronology violation.)

To be specific, consider the linear covariant Klein–Gordon equation

$$(\Box_{\boldsymbol{g}} + m^2)\phi = 0 \tag{1.1}$$

on a Lorentzian spacetime $(M, \boldsymbol{g})$. If $(M, \boldsymbol{g})$ is globally hyperbolic, there exists a (distributional) global advanced-minus-retarded fundamental solution $\Delta_{\boldsymbol{g}}(x, x')$ to (1.1) [6, 7] which plays a key rôle in the quantisation of real scalar field theory on $(M, \boldsymbol{g})$ via the commutation relation

$$[\phi(f_1), \phi(f_2)] = i\Delta_{\boldsymbol{g}}(f_1, f_2)\mathbb{1} \tag{1.2}$$

obeyed by smeared fields. Kay's proposal for field theory on more general spacetimes is that (1.2) should be replaced by a local equivalent in the following way. A candidate quantum field algebra for Eq. (1.1) is said to be *F-local* if each point in M has an intrinsically globally hyperbolic neighbourhood N such that (1.2) holds for test functions f_i supported in N and with $\Delta_{\boldsymbol{g}}$ replaced by $\Delta_{\boldsymbol{g}|_N}$, the advanced-minus-retarded fundamental solution for Eq. (1.1) on the globally hyperbolic spacetime $(N, \boldsymbol{g}|_N)$. If $(M, \boldsymbol{g})$ admits an F-local algebra for a given field theory it is said to be *F-quantum compatible* with that theory. (It is possible for a spacetime to be F-quantum compatible with massless Klein–Gordon theory but not with the massive theory [13].) All globally hyperbolic spacetimes are trivially F-quantum compatible with Klein–Gordon theory (for any mass) by virtue of the usual construction.

Kay argued in [1] that only an F-quantum compatible spacetime (or one obeying a similar condition) could arise as a semi-classical approximation to a state of quantum gravity. Thus the result proved in [1] that 2-dimensional Misner space fails to be F-quantum compatible, and the general 4-dimensional results proved by Kay, Radzikowski and Wald [9] showing that no spacetime containing a compactly generated Cauchy horizon can be F-quantum compatible may be taken as evidence that such spacetimes are unphysical. Further evidence for this view is provided by other results established in [9] which prove that, on a spacetime with compactly generated Cauchy horizon, the stress

energy tensor for any quantum 2-point function which is well-behaved (i.e., Hadamard) in the initial globally hyperbolic region must be ill-defined at certain points of the Cauchy horizon (see also [10, 11] for examples of this behaviour).

These results provide strong evidence in support of Hawking's *chronology protection conjecture* [8] that spacetimes with compactly generated Cauchy horizons (corresponding to the notion of a 'manufactured' time machine) are unphysical. Nonetheless there are many nonglobally hyperbolic spacetimes not covered by these results. The simplest cases are the 2- and 4-dimensional Minkowskian *spacelike cylinders* [1]; namely, 2- or 4-dimensional Minkowski space quotiented by a timelike translation. These spacetimes do not contain an initially globally hyperbolic region and therefore have no Cauchy horizon, thus avoiding the no-go theorems of [9]. In [1], Kay used Huygens' principle to show that the 4-dimensional spacelike cylinder is indeed F-quantum compatible with *massless* Klein–Gordon theory. Subsequently, the present author and Higuchi [12] used rather different methods to establish F-quantum compatibility of the 2- and 4-dimensional spacelike cylinders with massless and massive fields.[2] Indeed, the construction used showed that infinitely many F-local algebras exist on these spacetimes. These results demonstrated that the class of F-quantum compatible spacetimes is strictly larger than the class of globally hyperbolic spacetimes, but left open the question as to exactly how weak F-quantum compatibility is relative to global hyperbolicity.

Some progress in answering this question has now been made by the author, Higuchi and Kay, and will be described fully in [13]. It appears that the F-quantum compatibility of the 2- and 4-dimensional Minkowskian spacelike cylinders is unstable against certain classes of metric perturbations. The present paper is an outgrowth of [13] and consists of a full and rigorous treatment of the massive Klein–Gordon equation on 2-dimensional cylinder spacetimes.

To be specific, let the cylinder manifold M be the quotient of $\mathbb{R}^2$ with Cartesian coordinates (t, z) by the translation $z \mapsto z + 2\pi$. We consider a class of smooth metrics on M which agree with the Minkowski metric $\boldsymbol{\eta}$ outside a compact region K and are globally hyperbolic with respect to surfaces of constant t. Inside K, the metrics may differ greatly from $\boldsymbol{\eta}$. By choice of time orientation, the resulting spacetimes are either globally hyperbolic *timelike cylinders* (with z interpreted as a spatial coordinate) or nonglobally hyperbolic *spacelike cylinders* (with z interpreted as a temporal coordinate). Our principal results are that generic[3] timelike cylinders in our class admit a unique F-local algebra – the usual field algebra – for massive Klein–Gordon theory, but that generic spacelike cylinders are F-quantum incompatible with that theory. Indeed, similar results hold for the weaker notion of a *locally causal* algebra introduced in Section 3 which replaces Eq. (1.2) with commutativity of fields at local spacelike separation. We also refine our results to discuss cylinder spacetimes with some degree of z-translational invariance. For generic timelike cylinders in these classes we classify all possible F-local

[2]The 2-dimensional massless result was also known to Kay using different methods.

[3]We use this term in its topological sense: a subset of a topological space is *generic* if it contains a countable intersection of open dense sets (see, e.g., [14]).

algebras in terms of distributions on the symmetry group; generic spacelike cylinders still fail to be F-quantum compatible with massive Klein–Gordon theory.

These results show that F-locality does not provide significantly more freedom than is afforded by the usual globally hyperbolic theory on these 2-dimensional cylinder manifolds, at least for massive Klein–Gordon theory. It also appears that the F-quantum compatibility of the 2-dimensional Minkowskian spacelike cylinder with massive Klein–Gordon theory [12] is an anomalous case, presumably due to its full spacetime translational invariance. We have not found any other F-quantum compatible spacelike cylinder, and conjecture that there are in fact no others. The situation for massless fields and for 4-dimensional cylinder spacetimes is discussed in [13].

The structure of the paper is as follows. Section 2 contains some preliminaries relating to distributions on manifolds and causal structure, while Section 3 briefly reviews the algebraic approach to quantum field theory on curved spacetimes and Kay's F-locality proposal [1]. In addition we introduce the notion of a locally causal algebra. In Section 4 we define our class of timelike and spacelike cylinders and discuss various spaces of weak (bi)solutions to the Klein–Gordon equation on these spacetimes, and also develop a scattering formalism for this equation. Sections 5 and 6 form the main part of the paper. For generic metrics in our classes, we use a unique continuation result for Klein–Gordon bisolutions [15] and the scattering formalism to classify all bisolutions with the local causality property in Section 5. This leads into Section 6, in which we reduce questions concerning F-local and locally causal algebras to analogous questions concerning bisolutions with the corresponding properties, enabling the classification of all F-local and locally causal algebras on generic timelike and spacelike cylinders. We conclude with a discussion of our results in Section 7. There is one Appendix, in which we establish a property of the scattering operator employed in the main text.

2 Preliminaries

We begin by introducing the various spaces of test functions and distributions to be used in later sections. If M is any paracompact C^∞-manifold, $\mathcal{E}^p_q(M)$ will denote the space of smooth complex valued tensor fields of type (p,q) on M, equipped with its usual Fréchet space topology (see XVII.2 of Dieudonné [16]). In particular, we denote the space of scalar fields by $\mathcal{E}(M)$. Next, for any compact $K \subset M$, $\mathcal{D}(K)$ is defined to be the subspace of $\mathcal{E}(M)$ consisting of functions vanishing outside K. Each $\mathcal{D}(K)$ is again a Fréchet space. Choosing an increasing sequence K_m of compact subsets of M with $K_m \subset \operatorname{int} K_{m+1}$ and $M = \bigcup_{m\geq 1} K_m$, we define $\mathcal{D}(M)$ to be the strict inductive limit of the spaces $\mathcal{D}(K_m)$ (see, e.g., Sect. V.4 in [17]). The construction is independent of the particular sequence of compact sets used.

We define the space of distributions[4] $\mathcal{D}'(M)$ to be the topological dual of $\mathcal{D}(M)$; ele-

[4]Sometimes (as in §6.3 of [18]) the dual of $\mathcal{D}(M)$ is called the space of distribution densities, and the term distribution is used for the dual of the space of smooth compactly supported densities.

ments of $\mathcal{E}'(M)$, the topological dual of $\mathcal{E}(M)$, are precisely the distributions of compact support. We topologise $\mathcal{D}'(M)$ and $\mathcal{E}'(M)$ with their weak-$*$ topologies.

Note that smooth functions on M are not canonically embedded in $\mathcal{D}'(M)$. Instead, every positive density ρ on M defines an embedding $\iota : C^\infty(M) \to \mathcal{D}'(M)$ by

$$(\iota F)(f) = \int_M F f \rho \tag{2.1}$$

for all $f \in \mathcal{D}(M)$. In particular, if $\boldsymbol{g}$ is a metric on M, we will write $\iota_{\boldsymbol{g}}$ for the embedding corresponding to the density $|\det \boldsymbol{g}_{ab}|^{1/2}$.

A *bidistribution* on M is a bilinear, separately continuous functional $\Gamma(\cdot,\cdot)$ on $\mathcal{D}(M)\times\mathcal{D}(M)$. The space of bidistributions is denoted $\mathcal{D}^{(2)\prime}(M)$ and may be identified with $(\mathcal{D}(M)\otimes\mathcal{D}(M))'$ when $\mathcal{D}(M)\otimes\mathcal{D}(M)$ is given the projective tensor product topology (see §III.6 of Schaefer [19]). We endow $\mathcal{D}^{(2)\prime}(M)$ with the weak-$*$ topology arising from this identification, and also use the notations $\Gamma(f,g)$ and $\Gamma(f\otimes g)$ interchangeably. By the kernel theorem (XXIII.9 in [20]) $\mathcal{D}^{(2)\prime}(M)$ may also be identified with $\mathcal{D}'(M\times M)$.

Next, we briefly review some notions of causal structure (see, e.g., [21]). If M is a 2-dimensional C^∞-manifold, a Lorentzian (that is, signature zero) metric $\boldsymbol{g}$ on M is said to be *time-orientable* if there exists a global continuous choice (called a choice of *time orientation*) $p \mapsto C^+_{\boldsymbol{g},p} \subset T_pM$ of closed convex cones obtained at each $p\in M$ as the closure of a component of the set $\{\boldsymbol{v}\in T_pM \mid \boldsymbol{g}_p(\boldsymbol{v},\boldsymbol{v})\neq 0\}$ in the tangent space T_pM. In 2-dimensions there four possible local choices of time orientation at each point. Given a global time orientation $C^+_{\boldsymbol{g}}$, a smooth curve in M is said to be future directed and causal (resp., timelike) if its tangent vector at each point p on the curve lies in the local causal cone $C^+_{\boldsymbol{g},p}$ (resp., in the interior of $C^+_{\boldsymbol{g},p}$). The causal future (past) $J^+_{\boldsymbol{g}}(A)$ ($J^-_{\boldsymbol{g}}(A)$) of a subset $A\subset M$ with respect to $C^+_{\boldsymbol{g}}$ is the union of A with all points lying on future (past) directed causal curves from A. A *Cauchy surface* in M is a subset met exactly once by every inextendible timelike curve in M. If M admits a smooth foliation by C^∞-Cauchy surfaces, the triple $(M,\boldsymbol{g},C^+_{\boldsymbol{g}})$ is said to be *globally hyperbolic* [22]. A subset N of a time oriented spacetime $(M,\boldsymbol{g},C^+_{\boldsymbol{g}})$ is said to be *intrinsically globally hyperbolic* if $(N,\boldsymbol{g}|_N,C^+_{\boldsymbol{g}}|_N)$ is globally hyperbolic as a spacetime in its own right.

The *Klein–Gordon operator* $P_{\boldsymbol{g}}$ on $(M,\boldsymbol{g})$ is

$$P_{\boldsymbol{g}} = \Box_{\boldsymbol{g}} + \mu. \tag{2.2}$$

Here $\mu\in\mathbb{R}$, and the D'Alembertian $\Box_{\boldsymbol{g}}$ takes the form

$$\Box_{\boldsymbol{g}} = g^{-1/2}\partial_a g^{1/2}(\boldsymbol{g}^{-1})^{ab}\partial_b \tag{2.3}$$

in local coordinates, where $g = -\det\boldsymbol{g}_{ab}$ and $\boldsymbol{g}^{-1}$ is the (unique) inverse to $\boldsymbol{g}$, namely the smooth tensor field of type $(2,0)$ such that $(\boldsymbol{g}^{-1})^{ab}\boldsymbol{g}_{bc} = \delta^a{}_c$.[5] The operator $P_{\boldsymbol{g}}$ has

[5]Normally $(\boldsymbol{g}^{-1})^{ab}$ is written as $\boldsymbol{g}^{ab}$, but this notation is inappropriate when more than one metric is under consideration, as will be the case below.

the self-adjointness property

$$(\iota_{\boldsymbol{g}} f)(P_{\boldsymbol{g}} h) = (\iota_{\boldsymbol{g}} P_{\boldsymbol{g}} f)(h) \tag{2.4}$$

for $f, h \in \mathcal{D}(M)$, where $\iota_{\boldsymbol{g}}$ embeds $\mathcal{E}(M)$ in $\mathcal{D}'(M)$. If $(M, \boldsymbol{g}, C^+_{\boldsymbol{g}})$ is globally hyperbolic then [6, 7] there exist continuous maps $\Delta^{\pm}_{\boldsymbol{g}} : \mathcal{D}(M) \to \mathcal{E}(M)$ (the retarded $(+)$ and advanced $(-)$ Green functions for $P_{\boldsymbol{g}}$) such that for all $f \in \mathcal{D}(M)$,

$$P_{\boldsymbol{g}} \Delta^{\pm}_{\boldsymbol{g}} f = \Delta^{\pm}_{\boldsymbol{g}} P_{\boldsymbol{g}} f = f \tag{2.5}$$

and

$$\operatorname{supp} \Delta^{\pm}_{\boldsymbol{g}} f \subset J^{\pm}_{\boldsymbol{g}}(\operatorname{supp} f). \tag{2.6}$$

Furthermore, $\iota_{\boldsymbol{g}} \Delta^+_{\boldsymbol{g}} f$ $(\iota_{\boldsymbol{g}} \Delta^-_{\boldsymbol{g}} f)$, is the unique element φ of $\mathcal{D}'(M)$ having past (future) compact[6] support with respect to $C^+_{\boldsymbol{g}}$ and obeying $\varphi(P_{\boldsymbol{g}} h) = (\iota_{\boldsymbol{g}} f)(h)$ for all $h \in \mathcal{D}(M)$. The dual maps $\Delta^{\pm\prime}_{\boldsymbol{g}} : \mathcal{E}'(M) \to \mathcal{D}'(M)$ therefore obey $\Delta^{\pm\prime}_{\boldsymbol{g}} \iota_{\boldsymbol{g}} = \iota_{\boldsymbol{g}} \Delta^{\mp}_{\boldsymbol{g}}$.

The *advanced-minus-retarded fundamental solution* is the map $\Delta_{\boldsymbol{g}} : \mathcal{D}(M) \to \mathcal{E}(M)$ given by $\Delta_{\boldsymbol{g}} = \Delta^-_{\boldsymbol{g}} - \Delta^+_{\boldsymbol{g}}$. This map defines a bidistribution, also denoted $\Delta_{\boldsymbol{g}}$, by

$$\Delta_{\boldsymbol{g}}(f_1, f_2) = (\iota_{\boldsymbol{g}} \Delta_{\boldsymbol{g}} f_1)(f_2), \tag{2.7}$$

which is antisymmetric in f_1 and f_2 owing to the formula $\Delta'_{\boldsymbol{g}} \iota_{\boldsymbol{g}} = -\iota_{\boldsymbol{g}} \Delta_{\boldsymbol{g}}$ and is therefore a bisolution for $P_{\boldsymbol{g}}$, that is, $\Delta_{\boldsymbol{g}}(P_{\boldsymbol{g}} f_1, f_2) = \Delta_{\boldsymbol{g}}(f_1, P_{\boldsymbol{g}} f_2) = 0$ for all $f_i \in \mathcal{D}(M)$. In addition, the support properties of $\Delta^{\pm}_{\boldsymbol{g}}$ force $\Delta_{\boldsymbol{g}}(f_1, f_2)$ to vanish whenever the support of f_1 is spacelike separated from that of f_2 with respect to $C^+_{\boldsymbol{g}}$.

3 Locally Causal and F-local Algebras

In this section, we give a brief review of algebraic quantum field theory on globally hyperbolic curved spacetimes, and also describe Kay's F-locality condition. For a more detailed description see [1]. In addition, we introduce a weakened version of F-locality which we call 'local causality'.

Let M be a (2-dimensional) paracompact C^∞-manifold. The test functions $\mathcal{D}(M)$ label a set of abstract objects $\{\phi(f) \mid f \in \mathcal{D}(M)\}$ which will be interpreted as smeared fields. The $\phi(f)$ are used to generate a free $*$-algebra $\mathfrak{A}(M)$ over $\mathbb{C}$ and with $\mathbb{1}$, which is topologised in a natural way using the topology of $\mathcal{D}(M)$. In this topology, addition, multiplication, conjugation are continuous operations in $\mathfrak{A}(M)$, and $f \mapsto \phi(f)$ is continuous from $\mathcal{D}(M)$ to $\mathfrak{A}(M)$. We will describe any quotient of $\mathfrak{A}(M)$ by a closed (two-sided) $*$-ideal as a *$*$-algebra of smeared fields on M*. Thus these algebras consist of (congruence classes of) complex polynomials in the $\phi(f)$'s, their conjugates $\phi(f)^*$

[6] A set A is past (future) compact with respect to $C^+_{\boldsymbol{g}}$ if $J^-_{\boldsymbol{g}}(\{p\}) \cap A$ (resp., $J^+_{\boldsymbol{g}}(\{p\}) \cap A$) is compact for all $p \in M$.

and the identity $\mathbb{1}$. The topology has been chosen to ensure that if $\mathcal{A}$ is a $*$-algebra of smeared fields then its topological dual $\mathcal{A}'$ separates the points of $\mathcal{A}$.[7] The original development of F-locality [1] did not assume any topology on field algebras; here it will play a useful rôle in linking the distributional and algebraic aspects of the argument. In addition, smeared fields in [1, 12] were labelled by real-valued test functions rather than complex-valued test functions as here. This difference is in fact inessential for our purposes.

Now suppose that a Lorentzian metric $\boldsymbol{g}$ and time orientation $C^+_{\boldsymbol{g}}$ are specified so that $(M, \boldsymbol{g}, C^+_{\boldsymbol{g}})$ is globally hyperbolic. The *usual field algebra*, $\mathcal{A}(M, \boldsymbol{g})$, for real linear Klein–Gordon quantum field theory on $(M, \boldsymbol{g}, C^+_{\boldsymbol{g}})$ is defined to be the quotient of $\mathfrak{A}(M)$ by the following relations (which generate a closed $*$-ideal in $\mathfrak{A}(M)$):

(Q1) Hermiticity: $(\phi(f))^* = \phi(\overline{f})$ for all $f \in \mathcal{D}(M)$

(Q2) Linearity: $\phi(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1\phi(f_1) + \lambda_2\phi(f_2)$ for all $\lambda_i \in \mathbb{C}$, $f_i \in \mathcal{D}(M)$

(Q3) Field Equation: $\phi((\Box_{\boldsymbol{g}} + \mu)f) = 0$ for all $f \in \mathcal{D}(M)$.

(Q4) CCR's: $[\phi(f_1), \phi(f_2)] = i\Delta_{\boldsymbol{g}}(f_1, f_2)\mathbb{1}$ for all $f_i \in \mathcal{D}(M)$.

Relation (Q4) implements the quantisation of the theory, and also ensures that $\mathcal{A}(M, \boldsymbol{g})$ obeys the causality axiom of Wightman theory (see, e.g., [23]) namely, that $\phi(f_1)$ and $\phi(f_2)$ should commute whenever the supp f_i are spacelike separated with respect to $C^+_{\boldsymbol{g}}$.

If $(M, \boldsymbol{g}, C^+_{\boldsymbol{g}})$ is not globally hyperbolic this construction fails owing to the absence of a global advanced-minus-retarded fundamental bisolution. There are, of course, $*$-algebras of smeared fields on M obeying relations (Q1), (Q2) and (Q3), but these relations are not sufficient to specify a reasonable quantum field theory on $(M, \boldsymbol{g})$. In [1], Kay suggested the following *F-locality condition* as a possible substitute for relation (Q4).

Definition 3.1 (F-locality) *A $*$-algebra $\mathcal{A}$ of smeared fields on $(M, \boldsymbol{g}, C^+_{\boldsymbol{g}})$ obeying (Q1), (Q2) and (Q3) is said to be* F-local *(with respect to $C^+_{\boldsymbol{g}}$) if every point of M has an intrinsically globally hyperbolic neighbourhood N such that*

$$[\phi(f_1), \phi(f_2)] = i\Delta_{\boldsymbol{g}|_N}(f_1, f_2)\mathbb{1} \tag{3.1}$$

whenever the supp f_i *are contained in N, and where $\Delta_{\boldsymbol{g}|_N}$ is the advanced-minus-retarded fundamental bisolution on $(N, \boldsymbol{g}|_N, C^+_{\boldsymbol{g}}|_N)$. A spacetime $(M, \boldsymbol{g}, C^+_{\boldsymbol{g}})$ which admits an F-local algebra is said to be* F-quantum compatible *with real linear scalar field theory.*

F-locality thus requires that (Q4) should hold locally. Note that the time orientation plays an important rôle in this definition as it fixes the local form of commutators. Thus a spacetime $(M, \boldsymbol{g})$ which admits two global time orientations might be F-quantum compatible with respect to both, one or neither. (Of course, when these orientations are

[7] That is, $a_1 = a_2$ in $\mathcal{A}$ if and only if $\omega(a_1) = \omega(a_2)$ for all $\omega \in \mathcal{A}'$.

just mutual time reversals of each other then either both are F-quantum compatible, or neither is.)

One may consider further possible replacements for (Q4). For our purposes, the weaker notion of a *locally causal algebra* will be useful.

Definition 3.2 (Local causality) *A $*$-algebra $\mathcal{A}$ of smeared fields on $(M, \boldsymbol{g}, C_{\boldsymbol{g}}^{+})$ obeying (Q1), (Q2) and (Q3) is said to be* locally causal *(with respect to $C_{\boldsymbol{g}}^{+}$) if $\mathcal{A}$ is nonabelian and every point of M has an intrinsically globally hyperbolic neighbourhood N such that*

$$[\phi(f_1), \phi(f_2)] = 0 \tag{3.2}$$

whenever $\operatorname{supp} f_1$ *and* $\operatorname{supp} f_2$ *are spacelike separated from one another in N with respect to $C_{\boldsymbol{g}}^{+}$. A spacetime $(M, \boldsymbol{g}, C_{\boldsymbol{g}}^{+})$ which admits a locally causal algebra is said to be compatible with local causality.*

Local causality is just the requirement that Wightman causality should hold in a local fashion. We require that the algebra be nonabelian in order to exclude the abelian classical field algebra for Klein–Gordon theory. While we do not propose local causality as a sufficient condition for a quantum field algebra on $(M, \boldsymbol{g}, C_{\boldsymbol{g}}^{+})$, it is however a reasonable necessary condition, strictly weaker than F-locality.

It will be useful to make analogous definitions of local causality and F-locality for $P_{\boldsymbol{g}}$-bisolutions on M. In this paper we will say that a $P_{\boldsymbol{g}}$-bisolution Γ is F-local with respect to $C_{\boldsymbol{g}}^{+}$ if every point has an intrinsically globally hyperbolic neighbourhood N such that Γ agrees with $\Delta_{\boldsymbol{g}|N}$ on $N \times N$. The bisolution Γ will be called locally causal if each point has an intrinsically globally hyperbolic neighbourhood N such that Γ vanishes on pairs of test functions whose supports are spacelike separated in N with respect to $C_{\boldsymbol{g}}^{+}$. Neighbourhoods such as N will be called *neighbourhoods of F-locality* or *local causality* as appropriate. Note that an F-local (locally causal) bisolution must satisfy additional requirements (antisymmetry, reality, and nontriviality) in order to be the commutator of an F-local (locally causal) algebra.[8]

Let us observe that neither F-locality nor local causality assert any uniformity in the 'size' of the neighbourhoods of F-locality or local causality in which Eqs. (3.1) or (3.2) hold. In addition, the commutator $[\phi(f_1), \phi(f_2)]$ is not assumed to be a scalar multiple of the identity unless the supports of f_1 and f_2 are sufficiently small and close together. Remarkably, we will see that both these properties will hold (at least generically) on the 2-dimensional cylinder manifold.

4 Timelike and Spacelike Cylinders

[8] In [12] these additional requirements were included in the definition of an F-local bisolution.

4.1 Definitions

Here, we define the timelike and spacelike cylinders to be studied in later sections. The cylinder manifold M is the quotient of $\mathbb{R}^2$ by the translation $(t,z)\mapsto(t,z+2\pi)$, where (t,z) are Cartesian coordinates on $\mathbb{R}^2$. These coordinates induce smooth vector fields $\boldsymbol{t}$ and $\boldsymbol{z}$ on M by the formulae

$$(\boldsymbol{t}(f))\circ q=\frac{\partial}{\partial t}(f\circ q)\qquad\text{and}\qquad(\boldsymbol{z}(f))\circ q=\frac{\partial}{\partial z}(f\circ q),\tag{4.1}$$

for $f\in C^\infty(M)$, where $q:\mathbb{R}^2\to M$ is the defining quotient map. We also use q to define 'diamond' neighbourhoods $N_\epsilon(p)$ of each point $p\in M$ by

$$N_\epsilon(q(t_0,z_0))=\{q(t,z)\mid |t-t_0|+|z-z_0|<\epsilon\}\,,\tag{4.2}$$

and also to define the translations $T_{(\tau,\zeta)}$ on M by $T_{(\tau,\zeta)}:q(t,z)\mapsto q(t+\tau,z+\zeta)$. The translation acts on a function f on M by

$$\left((T_{(\tau,\zeta)}f)\circ q\right)(t,z)=(f\circ q)(t-\tau,z-\zeta).\tag{4.3}$$

Let $\mathcal{G}$ be the class of smooth, bounded Lorentzian metrics $\boldsymbol{g}$ on M with $\boldsymbol{g}(\boldsymbol{z},\boldsymbol{z})$ everywhere negative and bounded away from zero. In particular, $\mathcal{G}$ contains the Minkowskian metric $\boldsymbol{\eta}$ obeying $\boldsymbol{\eta}(\boldsymbol{t},\boldsymbol{t})=-\boldsymbol{\eta}(\boldsymbol{z},\boldsymbol{z})=1$ and $\boldsymbol{\eta}(\boldsymbol{t},\boldsymbol{z})=0$ at all points. Each $\boldsymbol{g}\in\mathcal{G}$ is globally time orientable in two ways (modulo time reversal) determined by the continuous cone fields $p\mapsto C^{T+}_{\boldsymbol{g},p}$ and $p\mapsto C^{S+}_{\boldsymbol{g},p}$ defined by

$$C^{T+}_{\boldsymbol{g},p}=\{\boldsymbol{v}\in T_p(M)\mid \boldsymbol{g}_p(\boldsymbol{v},\boldsymbol{v})\ge 0\text{ and }\boldsymbol{e}_p(\boldsymbol{v},\boldsymbol{t})\ge 0\}\tag{4.4}$$

and

$$C^{S+}_{\boldsymbol{g},p}=\{\boldsymbol{v}\in T_p(M)\mid \boldsymbol{g}_p(\boldsymbol{v},\boldsymbol{v})\le 0\text{ and }\boldsymbol{e}_p(\boldsymbol{v},\boldsymbol{z})\ge 0\},\tag{4.5}$$

where $\boldsymbol{e}$ is the background Euclidean metric defined by $\boldsymbol{e}(\boldsymbol{t},\boldsymbol{t})=\boldsymbol{e}(\boldsymbol{z},\boldsymbol{z})=1$ and $\boldsymbol{e}(\boldsymbol{t},\boldsymbol{z})=0$ at all points of M. We will refer to the triple $(M,\boldsymbol{g},C^{T+}_{\boldsymbol{g}})$ as a *timelike cylinder* and to $(M,\boldsymbol{g},C^{S+}_{\boldsymbol{g}})$ as a *spacelike cylinder*. In particular, $(M,\boldsymbol{\eta},C^{T+}_{\boldsymbol{\eta}})$ and $(M,\boldsymbol{\eta},C^{S+}_{\boldsymbol{\eta}})$ are the Minkowskian timelike and spacelike cylinders studied in [1, 12].[9] It is easy to show that all timelike cylinders are globally hyperbolic with Cauchy surfaces $q(\{t\}\times\mathbb{R})$. The corresponding advanced-minus-retarded solution $\Delta_{\boldsymbol{g}}$ will play an important rôle in our discussion of both timelike and spacelike cylinders. All the spacelike cylinders $(M,\boldsymbol{g},C^{S+}_{\boldsymbol{g}})$ are nonglobally hyperbolic because they contain closed timelike lines (e.g., the integral curves of $\boldsymbol{z}$, which are timelike with respect to $C^{S+}_{\boldsymbol{g}}$) but because they do not possess initially globally hyperbolic regions they evade the KRW theorems [9].

[9] In fact, the Minkowskian spacelike cylinder was defined in [1, 12] as the quotient of 2-dimensional Minkowski space with coordinates (t,z) by the timelike translation $t\mapsto t+T$ for some $T>0$ with the inherited causal structure. This is equivalent to our present definition under interchange of t and z in the case $T=2\pi$.

Let us note that the Klein–Gordon operator $P_{\boldsymbol{g}} = \Box_{\boldsymbol{g}} + \mu$ is locally hyperbolic on both timelike and spacelike cylinders. However, the interpretation of μ depends on the time orientation: on the timelike cylinder μ corresponds to the particle mass squared; on the spacelike cylinder μ corresponds to minus the particle mass squared. Our results in this paper hold for either sign of μ, but *not* for $\mu = 0$. In addition, we will assume that μ is not a negative integer to avoid unncessary technicalities.

Our results will apply to timelike and spacelike cylinders whose metrics agree with $\boldsymbol{\eta}$ outside a nonempty compact subset of M of the form $J^+_{\boldsymbol{\eta}}(\{p\}) \cap J^-_{\boldsymbol{\eta}}(\{p'\})$ for $p, p' \in M$ (using $C^{T+}_{\boldsymbol{\eta}}$ to define the causal future and past). Accordingly, we define $\mathcal{G}_K$ be the set of metrics in $\mathcal{G}$ which agree with $\boldsymbol{\eta}$ outside K. We will also discuss subclasses of $\mathcal{G}_K$ which exhibit some degree of z-translational symmetry. Let G stand for either $\mathbb{Z}_N$ ($N \geq 2$) or SO(2), where $\mathbb{Z}_N$ is realised as the subgroup of $\mathbb{R}/(2\pi\mathbb{Z})$ consisting of (equivalence classes of) $2\pi r/N$, for $r = 0, \ldots, N-1$, and SO(2) is realised as $\mathsf{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. The group G acts on M by the corresponding 'translations in z' $T_{(0,\zeta)}$ for $\zeta \in G$, and we use $\mathcal{G}^{(G)}_K$ to denote the set of metrics $\boldsymbol{g} \in \mathcal{G}_K$ which are G-invariant, namely those obeying

$$\boldsymbol{g} = T_{(0,\zeta)}\boldsymbol{g}, \tag{4.6}$$

for all $\zeta \in G$.

To conclude this section, we describe a topology on $\mathcal{G}_K$ and its subclasses. The topology is defined so that $\boldsymbol{g}_n \to \boldsymbol{g}$ in $\mathcal{G}_K$ if and only if $\boldsymbol{g}_n^{-1} \to \boldsymbol{g}^{-1}$ in $\mathcal{E}^2_0(M)$, the Fréchet space of tensor fields of type $(2,0)$. That is, $\boldsymbol{g}_n \to \boldsymbol{g}$ if and only if the components of $\boldsymbol{g}_n^{-1}$ converge to those of $\boldsymbol{g}^{-1}$ in all coordinate patches. The significance of this topology is that one may show (using energy norm arguments such as those in §7.4 of [24]) that the map

$$\boldsymbol{g} \mapsto \Delta_{\boldsymbol{g}} f \tag{4.7}$$

is continuous from $\mathcal{G}_K$ to $\mathcal{E}(M)$ for each fixed $f \in \mathcal{D}(M)$, and hence that matrix elements of an appropriate scattering operator vary continuously with the metric (see Sections 4.2 and Appendix A).

4.2 The Klein–Gordon Equation on M

We now describe various spaces of (bi)solutions for the Klein–Gordon operator $P_{\boldsymbol{g}}$ for $\boldsymbol{g} \in \mathcal{G}_K$, exploiting the global hyperbolicity of $(M, \boldsymbol{g}, C^{T+}_{\boldsymbol{g}})$ and the corresponding advanced-minus-retarded fundamental solution $\Delta_{\boldsymbol{g}}$. The three solution spaces considered here are the space of smooth solutions

$$\mathcal{F}_{\boldsymbol{g}} = \{u \in \mathcal{E}(M) \mid P_{\boldsymbol{g}} u = 0\}, \tag{4.8}$$

the space of weak solutions

$$\mathcal{W}_{\boldsymbol{g}} = \{\varphi \in \mathcal{D}'(M) \mid \varphi(P_{\boldsymbol{g}} f) = 0 \ \forall f \in \mathcal{D}(M)\}, \tag{4.9}$$

and the space of weak bisolutions

$$\mathcal{W}_{\boldsymbol{g}}^{(2)} = \{\Gamma \in \mathcal{D}^{(2)\prime}(M) \mid \Gamma(P_{\boldsymbol{g}} f_1, f_2) = \Gamma(f_1, P_{\boldsymbol{g}} f_2) = 0 \ \forall f_1, f_2 \in \mathcal{D}(M)\}. \tag{4.10}$$

We endow these spaces with the relative topologies inherited from $\mathcal{E}(M)$, $\mathcal{D}'(M)$ and $\mathcal{D}^{(2)\prime}(M)$ respectively. Since $(M, \boldsymbol{g}, C_{\boldsymbol{g}}^{T+})$ has compact Cauchy surfaces, we have $\mathcal{F}_{\boldsymbol{g}} = \mathrm{Ran}\, \Delta_{\boldsymbol{g}}$ and dually, $\mathcal{W}_{\boldsymbol{g}} = \mathrm{Ran}\, \Delta_{\boldsymbol{g}}'$ and $\mathcal{W}_{\boldsymbol{g}}^{(2)} = \mathrm{Ran}\, (\Delta_{\boldsymbol{g}} \otimes \Delta_{\boldsymbol{g}})'$.

Each weak solution $\varphi \in \mathcal{W}_{\boldsymbol{g}}$ defines a continuous linear functional $\langle \varphi; \cdot \rangle_{\boldsymbol{g}}$ on $\mathcal{F}_{\boldsymbol{g}}$ so that

$$\langle \varphi; \Delta_{\boldsymbol{g}} f \rangle_{\boldsymbol{g}} = \varphi(f) \tag{4.11}$$

for all $f \in \mathcal{D}(M)$, and the map $\varphi \mapsto \langle \varphi; \cdot \rangle_{\boldsymbol{g}}$ is in fact an isomorphism of the topological vector spaces $\mathcal{W}_{\boldsymbol{g}}$ and $\mathcal{F}_{\boldsymbol{g}}'$ when the latter space is given its weak-$*$ topology. Similarly, the map $\Gamma \mapsto \langle \Gamma; \cdot \rangle_{\boldsymbol{g}}^{(2)}$ given by

$$\langle \Gamma; \Delta_{\boldsymbol{g}} f_1 \otimes \Delta_{\boldsymbol{g}} f_2 \rangle_{\boldsymbol{g}}^{(2)} = \Gamma(f_1 \otimes f_2) \tag{4.12}$$

for $f_i \in \mathcal{D}(M)$ is an isomorphism of $\mathcal{W}_{\boldsymbol{g}}^{(2)}$ with $(\mathcal{F}_{\boldsymbol{g}} \otimes \mathcal{F}_{\boldsymbol{g}})'$.

The maps $\iota_{\boldsymbol{g}}$ and $\iota_{\boldsymbol{g}}^{(2)} = \iota_{\boldsymbol{g}} \otimes \iota_{\boldsymbol{g}}$ embed $\mathcal{F}_{\boldsymbol{g}}$ and $\mathcal{F}_{\boldsymbol{g}} \otimes \mathcal{F}_{\boldsymbol{g}}$ continuously and densely in $\mathcal{W}_{\boldsymbol{g}}$ and $\mathcal{W}_{\boldsymbol{g}}^{(2)}$. Defining $\sigma_{\boldsymbol{g}} : \mathcal{F}_{\boldsymbol{g}} \times \mathcal{F}_{\boldsymbol{g}} \to \mathbb{C}$ by

$$\sigma_{\boldsymbol{g}}(u, v) = \langle \iota_{\boldsymbol{g}} u; v \rangle_{\boldsymbol{g}} \tag{4.13}$$

we have $\sigma_{\boldsymbol{g}}(\Delta_{\boldsymbol{g}} f_1, \Delta_{\boldsymbol{g}} f_2) = \langle \iota_{\boldsymbol{g}} \Delta_{\boldsymbol{g}} f_1; \Delta_{\boldsymbol{g}} f_2 \rangle_{\boldsymbol{g}} = \Delta_{\boldsymbol{g}}(f_1, f_2)$. Thus $\sigma_{\boldsymbol{g}}$ is the usual symplectic form on $\mathcal{F}_{\boldsymbol{g}}$ and may be written in the familiar form

$$\sigma_{\boldsymbol{g}}(u, v) = \int_{\Sigma_t} \sqrt{h}\, (v \boldsymbol{n}(u) - u \boldsymbol{n}(v)) \tag{4.14}$$

for $u, v \in \mathcal{F}_{\boldsymbol{g}}$, where $\boldsymbol{n}$ is the unit vector field normal to $\Sigma_t = q(\{t\} \times [0, 2\pi])$, and $\sqrt{h} = |\boldsymbol{g}(\boldsymbol{z}, \boldsymbol{z})|^{1/2}$ is the density derived from the induced metric on Σ_t.

The Cauchy evolution from the 'in' region M^- to the 'out' region M^+ may be formulated as a scattering problem, and this will provide a useful viewpoint in Section 5. If $\boldsymbol{g} \in \mathcal{G}_K$ we define wave operators $\Omega_{\boldsymbol{g}}^{\pm} : \mathcal{F}_{\boldsymbol{\eta}} \to \mathcal{F}_{\boldsymbol{g}}$ so that for each $u \in \mathcal{F}_{\boldsymbol{\eta}}$, $\Omega_{\boldsymbol{g}}^{\pm} u$ is the unique element of $\mathcal{F}_{\boldsymbol{g}}$ agreeing with u in $M^{\mp}$. These maps are topological vector space isomorphisms with the symplectic property $\sigma_{\boldsymbol{g}}(\Omega_{\boldsymbol{g}}^{\pm} u_1, \Omega_{\boldsymbol{g}}^{\pm} u_2) = \sigma_{\boldsymbol{\eta}}(u_1, u_2)$ for all $u_i \in \mathcal{F}_{\boldsymbol{\eta}}$. The *scattering operator* $S_{\boldsymbol{g}} = (\Omega_{\boldsymbol{g}}^-)^{-1} \Omega_{\boldsymbol{g}}^+$ of $P_{\boldsymbol{g}}$ relative to $P_{\boldsymbol{\eta}}$ is therefore a symplectic topological isomorphism on $\mathcal{F}_{\boldsymbol{\eta}}$ with the property that $u^+ = S_{\boldsymbol{g}} u^-$ if and only if there exists $u \in \mathcal{F}_{\boldsymbol{g}}$ agreeing with $u^{\pm}$ in $M^{\pm}$.

The matrix elements of the scattering operator may be given explicitly in terms of $\Delta_{\boldsymbol{g}}$: if $f^{\pm} \in \mathcal{D}(M^{\pm})$ and $u = \Delta_{\boldsymbol{\eta}} f^-$ and $v = \Delta_{\boldsymbol{\eta}} f^+$, then $\Omega_{\boldsymbol{g}}^+ u = \Delta_{\boldsymbol{g}} f^-$, $\Omega_{\boldsymbol{g}}^- v = \Delta_{\boldsymbol{g}} f^+$ and

$$\sigma_{\boldsymbol{\eta}}(S_{\boldsymbol{g}} u, v) = \sigma_{\boldsymbol{g}}(\Omega_{\boldsymbol{g}}^+ u, \Omega_{\boldsymbol{g}}^- v) = \sigma_{\boldsymbol{g}}(\Delta_{\boldsymbol{g}} f^-, \Delta_{\boldsymbol{g}} f^+) = \Delta_{\boldsymbol{g}}(f^-, f^+). \tag{4.15}$$

The scattering operator $S_{\boldsymbol{g}}$ on $\mathcal{F}_{\boldsymbol{\eta}}$ extends continuously to a map (also denoted $S_{\boldsymbol{g}}$) on the space of weak solutions $\mathcal{W}_{\boldsymbol{\eta}}$. Similarly, we define a scattering operator $S_{\boldsymbol{g}}^{(2)}$ on $\mathcal{W}_{\boldsymbol{\eta}}^{(2)}$ by extending the map $S_{\boldsymbol{g}} \otimes S_{\boldsymbol{g}}$ on $\mathcal{F}_{\boldsymbol{\eta}} \otimes \mathcal{F}_{\boldsymbol{\eta}}$. Thus, $\Gamma^+ = S_{\boldsymbol{g}}^{(2)}\Gamma^-$ if and only if there is a unique weak bisolution Γ for $P_{\boldsymbol{g}}$ agreeing with Γ^- on $M^- \times M^-$ and with Γ^+ on $M^+ \times M^+$.

To conclude this section, we make the above discussion more explicit in the particular case of the flat metric $\boldsymbol{\eta}$ and where μ is neither zero nor a negative integer. Here, for any $f \in \mathcal{D}(M)$, $\Delta_{\boldsymbol{\eta}} f$ may be expressed as a $\mathcal{E}(M)$-convergent series

$$\Delta_{\boldsymbol{\eta}} f = \sum_{\substack{n\in\mathbb{Z}\\ \epsilon=\pm}} i\epsilon \xi_{n\epsilon}(f)\xi_{-n-\epsilon} \tag{4.16}$$

where the functions $\xi_{n\epsilon} \in \mathcal{E}(M)$ are given by

$$(\xi_{n\epsilon} \circ q)(t,z) = \frac{e^{-i\epsilon\omega_n t + inz}}{(4\pi\omega_n)^{1/2}} \tag{4.17}$$

with

$$\omega_n^{1/2} = \begin{cases} |n^2+\mu|^{1/4} & n^2 > -\mu \\ e^{i\pi/4}|n^2+\mu|^{1/4} & n^2 < -\mu. \end{cases} \tag{4.18}$$

In Eq. (4.16), and in what follows, we have denoted the distribution $\iota_{\boldsymbol{\eta}}\xi_{n\epsilon}$ by $\xi_{n\epsilon}$ to simplify the notation. Thus $\xi_{n\epsilon}(f)$ is shorthand for $(\iota_{\boldsymbol{\eta}}\xi_{n\epsilon})(f)$.

To prove (4.16) we employ the ($\mathcal{D}(M)$-convergent) Fourier expansion $f = \sum_{n\in\mathbb{Z}} f_n$, where $(f_n \circ q)(t,z) = e^{-inz}\alpha_n(t)$ with $\alpha_n \in C_0^\infty(\mathbb{R})$ and observe that

$$\left((\Delta_{\boldsymbol{\eta}}^{\pm} f_n)\circ q\right)(t,z) = e^{-inz}\int_{\mp\infty}^{t} dt'\,\alpha_n(t')\frac{\sin\omega_n(t-t')}{\omega_n}. \tag{4.19}$$

Hence $\Delta_{\boldsymbol{\eta}} f_n = \sum_{\epsilon=\pm} i\epsilon\xi_{n\epsilon}(f_n)\xi_{-n\,-\epsilon}$ from which (4.16) follows.

Equation (4.16) shows that the $\xi_{n\epsilon}$ form a basis for $\mathcal{F}_{\boldsymbol{\eta}}$ and also allows us to give analogous expansions for general elements of $\mathcal{W}_{\boldsymbol{\eta}}$ and $\mathcal{W}_{\boldsymbol{\eta}}^{(2)}$. For $\varphi \in \mathcal{W}_{\boldsymbol{\eta}}$ and $f \in \mathcal{D}(M)$ we have

$$\varphi(f) = \langle \varphi, \Delta_{\boldsymbol{\eta}} f\rangle_{\boldsymbol{\eta}} = \sum_{\substack{n\in\mathbb{Z}\\ \epsilon=\pm}} i\epsilon\langle\varphi,\xi_{-n-\epsilon}\rangle_{\boldsymbol{\eta}}\xi_{n\epsilon}(f) \tag{4.20}$$

by Eq. (4.11) so φ may be expressed as the $\mathcal{W}_{\boldsymbol{\eta}}$-convergent series

$$\varphi = \sum_{\substack{n\in\mathbb{Z}\\ \epsilon=\pm}} i\epsilon\langle\varphi,\xi_{-n-\epsilon}\rangle_{\boldsymbol{\eta}}\,\xi_{n\epsilon}, \tag{4.21}$$

in which the coefficients $\langle \varphi; \xi_{-n-\epsilon}\rangle_{\boldsymbol{\eta}}$ grow no faster than polynomially in n owing to the local regularity of φ. Applying (4.21) to the particular case $\varphi = \xi_{n'\epsilon'}$, we also obtain the formula

$$\sigma_{\boldsymbol{\eta}}(\xi_{n'\epsilon'}, \xi_{-n\,-\epsilon}) = \langle \xi_{n'\epsilon'}, \xi_{-n\,-\epsilon}\rangle_{\boldsymbol{\eta}} = -i\epsilon\delta_{nn'}\delta_{\epsilon\epsilon'} \tag{4.22}$$

for all n, n', ϵ, ϵ'. We will make use of this formula in Section 5.

In an exactly analogous fashion, any $\Gamma \in \mathcal{W}_{\boldsymbol{\eta}}^{(2)}$ may be expanded as

$$\Gamma = \sum_{\substack{n,n'\in\mathbb{Z}\\ \epsilon,\epsilon'=\pm}} -\epsilon\epsilon'\langle \Gamma; \xi_{-n-\epsilon}\otimes\xi_{-n'-\epsilon'}\rangle_{\boldsymbol{\eta}}^{(2)} \xi_{n\epsilon}\otimes\xi_{n'\epsilon'}, \tag{4.23}$$

with the series converging in $\mathcal{W}_{\boldsymbol{\eta}}^{(2)}$ and coefficients $\langle \Gamma; \xi_{-n-\epsilon}\otimes\xi_{-n'-\epsilon'}\rangle_{\boldsymbol{\eta}}^{(2)}$ growing no faster than polynomially in n, n'. In particular,

$$\Delta_{\boldsymbol{\eta}} = \sum_{\substack{n\in\mathbb{Z}\\ \epsilon=\pm}} i\epsilon\xi_{n\epsilon}\otimes\xi_{-n-\epsilon}. \tag{4.24}$$

5 Locally Causal Bisolutions on M

We now employ the formalism developed in Section 4.2 to classify locally causal bisolutions on the cylinder manifold. Our argument consists of two steps. First, consider a bisolution $\Gamma \in \mathcal{W}_{\boldsymbol{g}}^{(2)}$ ($\boldsymbol{g} \in \mathcal{G}_K$) which is locally causal with respect to $C_{\boldsymbol{g}}^{+}$ (standing for either $C_{\boldsymbol{g}}^{T+}$ or $C_{\boldsymbol{g}}^{S+}$). We will show that the local causality of Γ implies that the bisolutions $\Gamma^{\pm}$ for $P_{\boldsymbol{\eta}}$ with which Γ agrees in $M^{\pm}\times M^{\pm}$ are both translationally invariant, in the sense that

$$\Gamma^{\pm}(T_{(\tau,\zeta)}f_1 \otimes T_{(\tau,\zeta)}f_2) = \Gamma^{\pm}(f_1\otimes f_2) \tag{5.1}$$

for all $(\tau, \zeta) \in \mathbb{R}^2$. Second, noting that $\Gamma^{+} = S_{\boldsymbol{g}}^{(2)}\Gamma^{-}$, we use the scattering formalism of Section 4.2 to investigate the class of bisolutions in $\mathcal{W}_{\boldsymbol{\eta}}^{(2)}$ whose translational invariance is preserved by $S_{\boldsymbol{g}}^{(2)}$. For generic $\boldsymbol{g} \in \mathcal{G}_K^{(G)}$ we will show how this class may be parametrised by distributions on the symmetry group G. In particular, if G is trivial, the only possibility is for Γ to be a scalar multiple of $\Delta_{\boldsymbol{g}}$.

We begin by showing that local causality of Γ implies translational invariance of Γ^{+}. The proof for Γ^{-} is identical. Choose $\alpha > \pi/2$ and $p_1 \in M^{+}$ such that the set

$$H = \left\{T_{(\tau,\zeta)}p_1 \mid \tau \in [-\alpha, \alpha], \quad \zeta \in [0, 2\pi]\right\} \tag{5.2}$$

is contained in M^{+}. Because H is compact and Γ^{+} agrees with Γ in $H \times H$ we deduce the existence of an $\epsilon > 0$ such that for all $p \in H$ the diamond neighbourhood $N_\epsilon(p)$ is a neighbourhood of local causality for Γ. Now pick any p_2 spacelike separated from p_1 in $N_\epsilon(p_1)$. Since the causal structure of M^{+} is translationally invariant (due to invariance

of $\boldsymbol{\eta}$) $T_{(\tau,\zeta)}p_2$ is spacelike separated from $T_{(\tau,\zeta)}p_1$ in the $N_\epsilon(T_{(\tau,\zeta)}p_1)$ for all $\tau \in [-\alpha, \alpha]$, $\zeta \in [0, 2\pi]$. Thus Γ^+ must vanish in a neighbourhood of

$$S = \left\{ (T_{(\tau,\zeta)}p_1, T_{(\tau,\zeta)}p_2) \mid \tau \in [-\pi/2, \pi/2], \quad \zeta \in [0, 2\pi] \right\} \tag{5.3}$$

in $M \times M$ because each $N_\epsilon(T_{(\tau,\zeta)}p_1)$ is a neighbourhood of local causality for Γ. We now apply the following result to conclude that $\Gamma^\pm$ are translationally invariant.

Theorem 5.1 *If μ is neither zero nor a negative integer and Γ_0 is a weak $P_{\boldsymbol{\eta}}$-bisolution vanishing on an open neighbourhood of a surface of the form (5.3), then Γ_0 is translationally invariant.*

Theorem 5.1 is proved in [15] using methods drawn from Beurling's theory of interpolation [25]. The result does *not* hold if $\mu = 0$ (the requirement that μ is not a negative integer is included for technical convenience). Now translational invariance of a bisolution $\Gamma_0 \in \mathcal{W}_{\boldsymbol{\eta}}^{(2)}$ entails that in the expansion

$$\Gamma_0 = \sum_{\substack{n,n' \in \mathbb{Z} \\ \epsilon,\epsilon' = \pm}} \gamma_{nn'}^{\epsilon\epsilon'} \xi_{n\epsilon} \otimes \xi_{n'\epsilon'}, \tag{5.4}$$

all coefficients vanish except those on the anti-diagonal $n = -n'$, $\epsilon = -\epsilon'$. Accordingly, the bisolutions $\Gamma^\pm$ may be expanded in the form

$$\Gamma^- = \sum_{\substack{n \in \mathbb{Z} \\ \epsilon = \pm}} i\epsilon \gamma_{n\epsilon} \xi_{n\epsilon} \otimes \xi_{-n\,-\epsilon} \tag{5.5}$$

and

$$\Gamma^+ = \sum_{\substack{n \in \mathbb{Z} \\ \epsilon = \pm}} i\epsilon \widetilde{\gamma}_{n\epsilon} \xi_{n\epsilon} \otimes \xi_{-n\,-\epsilon} \tag{5.6}$$

with polynomially bounded coefficients $\gamma_{n\epsilon}$ and $\widetilde{\gamma}_{n\epsilon}$. But we also have

$$\Gamma^+ = S_{\boldsymbol{g}}^{(2)} \Gamma^- = \sum_{\substack{n \in \mathbb{Z} \\ \epsilon = \pm}} i\epsilon \gamma_{n\epsilon} (S_{\boldsymbol{g}} \xi_{n\epsilon}) \otimes (S_{\boldsymbol{g}} \xi_{-n\,-\epsilon}). \tag{5.7}$$

On the one hand, therefore,

$$\begin{aligned} \langle \Gamma^+; S_{\boldsymbol{g}} \xi_{-n'\,-\epsilon'}, \xi_{n''\epsilon''} \rangle &= \sum_{\substack{n \in \mathbb{Z} \\ \epsilon = \pm}} i\epsilon \widetilde{\gamma}_{n\epsilon} \sigma_{\boldsymbol{\eta}}(\xi_{n\epsilon}, S_{\boldsymbol{g}} \xi_{-n'\,-\epsilon'}) \sigma_{\boldsymbol{\eta}}(\xi_{-n\,-\epsilon}, \xi_{n''\epsilon''}) \\ &= \widetilde{\gamma}_{n''\epsilon''} \sigma_{\boldsymbol{\eta}}(S_{\boldsymbol{g}} \xi_{-n'\,-\epsilon'}, \xi_{n''\epsilon''}) \end{aligned} \tag{5.8}$$

by Eqs. (5.6) and (4.22) and antisymmetry of $\sigma_{\boldsymbol{g}}$, whilst on the other hand

$$\begin{aligned} \langle \Gamma^+; S_{\boldsymbol{g}} \xi_{-n'\,-\epsilon'}, \xi_{n''\epsilon''} \rangle &= \sum_{\substack{n \in \mathbb{Z} \\ \epsilon = \pm}} i\epsilon \gamma_{n\epsilon} \sigma_{\boldsymbol{\eta}}(S_{\boldsymbol{g}} \xi_{n\epsilon}, S_{\boldsymbol{g}} \xi_{-n'\,-\epsilon'}) \sigma_{\boldsymbol{\eta}}(S_{\boldsymbol{g}} \xi_{-n\,-\epsilon}, \xi_{n''\epsilon''}) \\ &= \gamma_{n'\epsilon'} \sigma_{\boldsymbol{\eta}}(S_{\boldsymbol{g}} \xi_{-n'\,-\epsilon'}, \xi_{n''\epsilon''}) \end{aligned} \tag{5.9}$$

using Eqs. (5.7) and (4.22) and the symplectic property of $S_{\boldsymbol{g}}$. Thus

$$\widetilde{\gamma}_{n''\epsilon''}\sigma_{\boldsymbol{\eta}}(S_{\boldsymbol{g}}\xi_{-n'\,-\epsilon'},\xi_{n''\epsilon''}) = \sigma_{\boldsymbol{\eta}}(S_{\boldsymbol{g}}\xi_{-n'\,-\epsilon'},\xi_{n''\epsilon''})\gamma_{n'\epsilon'} \tag{5.10}$$

for all $n',\epsilon',n'',\epsilon''$. We now invoke the following result, which will be proved in Appendix A.

Theorem 5.2 *For generic $\boldsymbol{g}\in\mathcal{G}_K^{(G)}$, we have $\sigma_{\boldsymbol{\eta}}(S_{\boldsymbol{g}}\xi_{-n-\epsilon},\xi_{n'\epsilon'})\neq 0$ whenever $n=n'$ (if $G=\mathrm{SO}(2)$) or $n\equiv n' \pmod N$ (if $G=\mathbb{Z}_N$).*[10]

Thus, if $\boldsymbol{g}$ belongs to the generic class marked out by Theorem 5.2, we deduce from Eq. (5.10) that $\gamma_{n\epsilon}=\widetilde{\gamma}_{n'\epsilon'}$ whenever $n\equiv n' \pmod N$ (if $G=\mathbb{Z}_N$) or $n=n'$ (if $G=\mathrm{SO}(2)$). Thus $\gamma_{n+}=\gamma_{n-}=\widetilde{\gamma}_{n+}=\widetilde{\gamma}_{n-}=\gamma_n$ for all $n\in\mathbb{Z}$ with $\gamma_n=\gamma_{n'}$ whenever $n\equiv n' \pmod N$ if $G=\mathbb{Z}_N$. Reassembling $\Gamma^\pm$ we have

$$\Gamma^+=\Gamma^-=\sum_{\substack{n\in\mathbb{Z}\\ \epsilon=\pm}} i\epsilon\gamma_n\xi_{n\epsilon}\otimes\xi_{-n\,-\epsilon}. \tag{5.11}$$

The final step in our argument here is to write $\Gamma^\pm$ in terms of a distribution on G. Here, we define the test functions $\mathcal{D}(G)$ to be the set of all complex valued functions on G if $G=\mathbb{Z}_N$, and use the usual space of test functions $\mathcal{D}(\mathsf{S}^1)$ if $G=\mathrm{SO}(2)$. Now for any $f_1,f_2\in\mathcal{D}(M)$ the formula

$$(f_1\diamond_{\boldsymbol{g}} f_2)(\zeta)=\Delta_{\boldsymbol{g}}(T_{(0,\zeta)}f_1,f_2) \tag{5.12}$$

defines a function $f_1\diamond_{\boldsymbol{g}} f_2 : G\to\mathbb{C}$ belonging to $\mathcal{D}(G)$. In particular, if $f_i\in\mathcal{D}(M^-)$ we have

$$(f_1\diamond_{\boldsymbol{g}} f_2)(\zeta)=\Delta_{\boldsymbol{\eta}}(T_{(0,\zeta)}f_1,f_2)=\sum_{\substack{n\in\mathbb{Z}\\ \epsilon=\pm}} i\epsilon e^{in\zeta}\xi_{n\epsilon}(f_1)\xi_{-n\,-\epsilon}(f_2), \tag{5.13}$$

since $\boldsymbol{g}=\boldsymbol{\eta}$ in M^-. We claim that there exists a unique $\psi\in\mathcal{D}'(G)$ obeying $\psi(e_n)=\gamma_n$ for all $n\in\mathbb{Z}$, where $e_n(\zeta)=e^{in\zeta}$. This is evident if $G=\mathrm{SO}(2)$ because the γ_n are polynomially bounded. In the case $G=\mathbb{Z}_N$, we note that $n\equiv n' \pmod N$ entails that $\gamma_n=\gamma_{n'}$ and also that e_n and $e_{n'}$ agree as functions on G, so the conditions $\psi(e_n)=\gamma_n$ constitute N independent conditions on the N-dimensional space $\mathcal{D}'(G)$. Comparing with Eq. (5.11) we observe that

$$\Gamma^-(f_1,f_2)=\psi(f_1\diamond_{\boldsymbol{g}} f_2). \tag{5.14}$$

Now the right hand side of this formula defines a global bisolution Γ_ψ for $P_{\boldsymbol{g}}$ because G-invariance of $\boldsymbol{g}$ entails that $h_1\diamond_{\boldsymbol{g}}(P_{\boldsymbol{g}}h_2)=(P_{\boldsymbol{g}}h_1)\diamond_{\boldsymbol{g}} h_2=0$, the zero function on G, for any $h_i\in\mathcal{D}(M)$. Since Γ agrees with Γ_ψ in $M^-\times M^-$, we have $\Gamma=\Gamma_\psi$ identically.

We may summarise the discussion above in the following statement.

[10]In the case $N=1$, $n\equiv n' \pmod N$ for all $n,n'\in\mathbb{Z}$.

Theorem 5.3 *For generic $\boldsymbol{g} \in \mathcal{G}_K^{(G)}$ a bisolution $\Gamma \in \mathcal{W}_{\boldsymbol{g}}^{(2)}$ is locally causal with respect to $C_{\boldsymbol{g}}^{T+}$ or $C_{\boldsymbol{g}}^{S+}$ only if $\Gamma = \Gamma_\psi$ for some $\psi \in \mathcal{D}'(G)$, where $\Gamma_\psi(f_1, f_2) = \psi(f_1 \diamond_{\boldsymbol{g}} f_2)$.*

In particular, consider the case in which G is trivial, $G = \{0\}$. Here $\mathcal{D}'(G)$ is the 1-dimensional space of scalar multiples of δ_0, the δ-distribution at $0 \in G$. Now if $\psi = \lambda\delta_0$ then $\psi(f_1 \diamond_{\boldsymbol{g}} f_2) = \lambda\Delta_{\boldsymbol{g}}(f_1, f_2)$ by Eq. (5.12), so Theorem 5.3 asserts that Γ is locally causal with respect to $C_{\boldsymbol{g}}^{T+}$ or $C_{\boldsymbol{g}}^{T+}$ only if Γ is a scalar multiple of $\Delta_{\boldsymbol{g}}$. Furthermore, since $\Delta_{\boldsymbol{g}}$ is not locally causal with respect to $C_{\boldsymbol{g}}^{S+}$ we conclude that there are no nontrivial locally causal bisolutions on generic spacelike cylinders with metrics in $\mathcal{G}_K$.

6 Quantum Field Theory on Cylinder Spacetimes

In this section we will state and prove our results concerning locally causal and F-local algebras on the timelike and spacelike cylinders with metrics belonging to our classes $\mathcal{G}_K$ or $\mathcal{G}_K^{(G)}$ for $G = \mathbb{Z}_N$ or SO(2). The essential idea is to reduce questions concerning locally causal and F-local algebras to analogous questions concerning the corresponding bisolutions, which can then be answered using the results of Section 5. We will only address the case in which the mass parameter μ in the Klein–Gordon operator is neither zero nor a negative integer, i.e., $\mu \notin \mathbb{Z}^-$.

Let $\boldsymbol{g} \in \mathcal{G}_K$ and suppose that $\mathcal{A}$ is a locally causal $*$-algebra of smeared fields on $(M, \boldsymbol{g}, C_{\boldsymbol{g}}^+)$ where $C_{\boldsymbol{g}}^+$ stands for either $C_{\boldsymbol{g}}^{T+}$ or $C_{\boldsymbol{g}}^{S+}$. If ω is any element of the dual $\mathcal{A}'$, we may define a bilinear functional Γ on $\mathcal{D}(M) \times \mathcal{D}(M)$ by

$$\Gamma(f_1, f_2) = \omega([\phi(f_1), \phi(f_2)]). \tag{6.1}$$

The functional Γ is a bidistribution owing to the topology of $\mathcal{A}$; it is also a locally causal bisolution for $P_{\boldsymbol{g}}$ by the field equation (Q3) and local causality of $\mathcal{A}$. If $\boldsymbol{g}$ belongs to the generic subset of $\mathcal{G}_K^{(G)}$ demarcated by Theorem 5.3 we may deduce that $\Gamma = \Gamma_{\psi_\omega}$ for some $\psi_\omega \in \mathcal{D}'(G)$, so

$$\omega([\phi(f_1), \phi(f_2)]) = i\psi_\omega(f_1 \diamond_{\boldsymbol{g}} f_2) \tag{6.2}$$

for all $f_i \in \mathcal{D}(M)$. Since ω was arbitrary and $\mathcal{A}'$ separates the points of $\mathcal{A}$, the commutator $[\phi(f_1), \phi(f_2)]$ can therefore depend on f_1 and f_2 only through the combination $f_1 \diamond_{\boldsymbol{g}} f_2$, so

$$[\phi(f_1), \phi(f_2)] = i\Psi(f_1 \diamond_{\boldsymbol{g}} f_2) \tag{6.3}$$

for some linear $\Psi : \mathcal{D}(G) \to \mathcal{A}$. The key step is provided by the next Lemma.

Lemma 6.1 $\Psi(\cdot) = \psi(\cdot)\mathbb{1}$ *for some $\psi \in \mathcal{D}'(G)$.*

Lemma 6.1 will be proved using a Jacobi identity argument at the end of this section. It follows that for generic $\boldsymbol{g} \in \mathcal{G}_K^{(G)}$ the commutator in any locally causal algebra on $(M, \boldsymbol{g}, C_{\boldsymbol{g}}^{+})$ takes the form

$$[\phi(f_1), \phi(f_2)] = i\psi(f_1 \diamond_{\boldsymbol{g}} f_2)\mathbb{1} = i\Gamma_\psi(f_1, f_2)\mathbb{1} \tag{6.4}$$

for some $\psi \in \mathcal{D}'(G)$ depending on $\mathcal{A}$. In the simplest case, where G is trivial, we obtain the following.

Theorem 6.2 *Suppose $\mu \notin \mathbb{Z}^-$. For generic $\boldsymbol{g} \in \mathcal{G}_K$, we have: (1) $\mathcal{A}$ is locally causal on $(M, \boldsymbol{g}, C_{\boldsymbol{g}}^{T+})$ if and only if $\mathcal{A} = \mathcal{A}_\lambda(M, \boldsymbol{g})$, the quotient of $\mathfrak{A}(M)$ by (Q1–3) and*

(Q4)$_\lambda$ CCR's: $[\phi(f_1), \phi(f_2)] = i\lambda\Delta_{\boldsymbol{g}}(f_1, f_2)\mathbb{1}$ for all $f_i \in C_0^\infty(M; \mathbb{R})$

for some $\lambda \in \mathbb{R}\backslash\{0\}$. (2) There is no locally causal algebra on $(M, \boldsymbol{g}, C_{\boldsymbol{g}}^{S+})$.

Proof: Assume that $\boldsymbol{g}$ belongs to the generic class of Theorem 5.3.

(1) The commutator takes the form (6.4) for some $\psi \in \mathcal{D}'(G)$. Since $G = \{0\}$, we have $\Gamma_\psi = \lambda\Delta_{\boldsymbol{g}}$ for some $\lambda \in \mathbb{C}$ by the discussion at the end of Section 5, so $[\phi(f_1), \phi(f_2)] = i\lambda\Delta_{\boldsymbol{g}}(f_1, f_2)\mathbb{1}$ for all $f_i \in \mathcal{D}(M)$. We deduce that λ is real due to the antihermiticity of the commutator, and since $\mathcal{A}$ is nonabelian we must have $\lambda \neq 0$. Thus $\mathcal{A}$ satisfies relations (Q1–3) and (Q4)$_\lambda$ and must be a quotient of $\mathcal{A}_\lambda(M, \boldsymbol{g})$. But $\mathcal{A}_\lambda(M, \boldsymbol{g})$ is simple (see, e.g., §7.1 of [26]), so $\mathcal{A} = \mathcal{A}_\lambda(M, \boldsymbol{g})$.

(2) If a locally causal algebra existed, its commutator would be proportional to $\Delta_{\boldsymbol{g}}$. Since this is not locally causal with respect to $C_{\boldsymbol{g}}^{S+}$, the proportionality constant would vanish and the algebra would be abelian – a contradiction. ■

Corollary 6.3 *Suppose $\mu \notin \mathbb{Z}^-$. For generic $\boldsymbol{g} \in \mathcal{G}_K$, we have: (1) The usual field algebra $\mathcal{A}(M, \boldsymbol{g})$ is the unique F-local algebra on $(M, \boldsymbol{g}, C_{\boldsymbol{g}}^{T+})$. (2) The spacelike cylinder $(M, \boldsymbol{g}, C_{\boldsymbol{g}}^{S+})$ is F-quantum incompatible.*

Proof: (1) Comparing (Q4)$_\lambda$ with the F-locality condition, we require $\lambda = 1$ for F-locality. (2) This is immediate from Theorem 6.2 because any F-local algebra is necessarily locally causal.

These (perhaps surprising) results show that F-locality and local causality are much stronger conditions than might have been expected on these spacetimes. For generic (globally hyperbolic) timelike cylinders, one is essentially restricted to the usual algebra even in the locally causal case, because $\mathcal{A}_\lambda(M, \boldsymbol{g})$ is related to $\mathcal{A}(M, \boldsymbol{g})$ by rescaling Planck's constant by a factor of $|\lambda|$ and a change of time orientation from $C_{\boldsymbol{g}}^{T+}$ to $-C_{\boldsymbol{g}}^{T+}$ if $\lambda < 0$.

The second parts of Theorem 6.2 and Corollary 6.3 should be set against the fact that the Minkowskian spacelike cylinder $(M, \boldsymbol{\eta}, C_{\boldsymbol{\eta}}^{S+})$ admits infinitely many F-local algebras

for massive (and massless) Klein–Gordon theory [12]. It is clear we cannot replace 'generic' by 'all' in the statements of these results, which raises the issue as to whether there is a substantial – albeit nongeneric in $\mathcal{G}_K$ – class of metrics whose corresponding spacelike cylinders are F-quantum compatible. To address this, we refine the above results to cover generic G-invariant metrics belonging to $\mathcal{G}_K^{(G)}$ for $G = \mathbb{Z}_N$ ($N \geq 2$) or SO(2). These spaces are of course nongeneric subsets of $\mathcal{G}_K$. Because the commutator has already been established in Eq. (6.4), it remains to determine more precisely the class of $\psi \in \mathcal{D}'(G)$ for which Γ_ψ can determine commutators in a locally causal or F-local algebra. We begin by noting that ψ must be nontrivial because such algebras are nonabelian. Next, ψ is constrained by the algebraic relations

$$[\phi(f_1), \phi(f_2)] = -[\phi(f_2), \phi(f_1)] \qquad \text{and} \qquad [\phi(f_1), \phi(f_2)]^* = -[\phi(\overline{f_1}), \phi(\overline{f_2})] \tag{6.5}$$

which follow from (Q1) and imply that $\Gamma_\psi(f_1, f_2) = -\Gamma_\psi(f_2, f_1)$ and $\overline{\Gamma_\psi(f_1, f_2)} = \Gamma_\psi(\overline{f_1}, \overline{f_2})$. In consequence, ψ is *even*, i.e., $\psi(\widetilde{\gamma}) = \psi(\gamma)$ for $\gamma \in \mathcal{D}(G)$, where $\widetilde{\gamma}(\zeta) = \gamma(-\zeta)$ for all $\zeta \in G$, and *real*, i.e., $\overline{\psi(\gamma)} = \psi(\overline{\gamma})$.

The remaining requirement is that Γ_ψ should be locally causal or F-local with respect to $C_{\boldsymbol{g}}^+$. In the case $G = \mathbb{Z}_N$, $\psi \in \mathcal{D}'(G)$ may be regarded as a vector $\psi = (\psi_0, \psi_1, \ldots, \psi_{N-1}) \in \mathbb{C}^N$ with action

$$\psi(\gamma) = \sum_{r=0}^{N-1} \psi_r \gamma(2\pi r/N) \tag{6.6}$$

on a function $\gamma : \mathbb{Z}_N \to \mathbb{C}$. [Recall that $\mathbb{Z}_N$ is realised as the group of equivalence classes of $2\pi r/N$, $r = 0, \ldots, N-1$.] Thus

$$\Gamma_\psi(f_1, f_2) = \sum_{r=0}^{N-1} \psi_r \Delta_{\boldsymbol{g}}(T_{(0,2\pi r/N)} f_1, f_2), \tag{6.7}$$

and if the f_i are supported within a common sufficiently small neighbourhood only the $r = 0$ term in Eq. (6.7) contributes yielding $\Gamma_\psi(f_1, f_2) = \psi_0 \Delta_{\boldsymbol{g}}(f_1, f_2)$. Thus Γ_ψ is automatically locally causal with respect to $C_{\boldsymbol{g}}^{T+}$, and is F-local with respect to $C_{\boldsymbol{g}}^{T+}$ if and only if $\psi_0 = 1$. On the other hand, Γ_ψ is locally causal with respect to $C_{\boldsymbol{g}}^{S+}$ if and only if $\psi_0 = 0$ and can never be F-local with respect to this time orientation. A more involved analysis for SO(2) (which we will omit) leads to the conclusions that Γ_ψ is locally causal with respect to $C_{\boldsymbol{g}}^{T+}$ if and only if condition

(T) ψ vanishes on $W \backslash \{0\}$ for some neighbourhood W of $0 \in G$

holds; that Γ_ψ is F-local with respect to $C_{\boldsymbol{g}}^{T+}$ if and only if condition

(T′) ψ agrees with δ_0, the δ-distribution at $0 \in G$, in some neighbourhood W of $0 \in G$

holds; and that Γ_ψ is locally causal with respect to $C_{\boldsymbol{g}}^{S+}$ if and only if condition

(S) ψ vanishes on some neighbourhood W of $0 \in G$

holds. In this last case, Γ_ψ vanishes on *all* pairs of test functions supported in sufficiently small common neighbourhoods, so no bisolution of the form Γ_ψ can be F-local with respect to $C^{S+}_{\boldsymbol{g}}$. We note that the conditions (T), (T′) and (S) coincide with the conditions given on ψ_0 in the case $G = \mathbb{Z}_N$ if one endows G with the discrete topology (so that $\{0\}$ is a neighbourhood of 0). In particular, condition (T) is trivially satisfied.

The above discussion leads immediately to the following conclusions.

Theorem 6.4 *Suppose $\mu \notin \mathbb{Z}^-$. For generic $\boldsymbol{g} \in \mathcal{G}^{(G)}_K$ we have: (1) An algebra $\mathcal{A}$ of smeared fields is locally causal on the timelike cylinder $(M, \boldsymbol{g}, C^{T+}_{\boldsymbol{g}})$ if and only if the commutation relation*

(Q4)$_\psi$ Modified CCR's: $[\phi(f_1), \phi(f_2)] = i\Gamma_\psi(f_1, f_2)\mathbb{1}$ for all $f_i \in C^\infty_0(M;\mathbb{R})$

holds for some nontrivial, real, even $\psi \in \mathcal{D}'(G)$ obeying condition (T) above. (2) An algebra $\mathcal{A}$ of smeared fields is locally causal on the spacelike cylinder $(M, \boldsymbol{g}, C^{S+}_{\boldsymbol{g}})$ if and only if the conditions of part (1) hold with condition (T) replaced by condition (S).

Corollary 6.5 *Suppose $\mu \notin \mathbb{Z}^-$. For generic $\boldsymbol{g} \in \mathcal{G}^{(G)}_K$ we have: (1) An algebra $\mathcal{A}$ of smeared fields is F-local on the timelike cylinder $(M, \boldsymbol{g}, C^{T+}_{\boldsymbol{g}})$ if and only if the conditions of part (1) of Theorem 6.4 hold with condition (T) replaced by condition (T′). (2) The spacelike cylinder $(M, \boldsymbol{g}, C^{S+}_{\boldsymbol{g}})$ is F-quantum incompatible.*

In fact, the locally causal algebras described part (2) of Theorem 6.4 are ‘locally abelian’ (due to condition (S)) and are not serious candidates for a quantum field algebra on the spacelike cylinder.

To be more quantitative concerning the range of F-local algebras admitted by a generic $\mathbb{Z}_N$-invariant timelike cylinder, we note that $\psi = (\psi_0, \ldots \psi_{N-1}) \in \mathcal{D}'(\mathbb{Z}_N)$ is real if and only if each $\psi_r \in \mathbb{R}$; even if and only if $\psi_r = \psi_{N-r}$ for $r = 1, \ldots, N-1$; and F-local if and only if $\psi_0 = 1$. Regarding $\mathcal{D}'(\mathbb{Z}_N)$ as a $2N$-dimensional real vector space, the F-local elements lie on a $[\frac{1}{2}N]$-dimensional hyperplane, where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

Corollary 6.5 shows that the F-quantum compatible Minkowskian spacelike cylinder $(M, \boldsymbol{\eta}, C^{S+}_{\boldsymbol{g}})$ (see [12]) represents an extremely special case even within classes of metrics exhibiting a high degree of symmetry. We know of no metric other than $\boldsymbol{\eta}$ in $\mathcal{G}_K$ whose corresponding spacelike cylinder is F-quantum compatible with massive Klein–Gordon theory, and conjecture that no such metric exists. The F-quantum compatibility of $(M, \boldsymbol{\eta}, C^{S+}_{\boldsymbol{g}})$ appears to rely in an essential way on its invariance under all spacetime translations.

To summarise our results in this section, we have shown that local causality is a highly restrictive condition on the 2-dimensional cylinder manifold. Generically in $\mathcal{G}_K$,

with no symmetry assumed, we have essentially no more freedom than is afforded by the usual construction of quantum field theory in curved spacetime: there is a 1-parameter family of locally causal algebras $\mathcal{A}_\lambda(M,\boldsymbol{g})$ on the globally hyperbolic timelike cylinder of which exactly one is F-local, and the nonglobally hyperbolic spacelike cylinder fails to be compatible with local causality. By invoking symmetry, some degree of freedom is obtained and we have classified the resulting locally causal and F-local algebras for generic timelike cylinders arising from the classes $\mathcal{G}_K^{(G)}$ ($G=\mathbb{Z}_N$ or SO(2)). However the spacelike cylinders remain F-quantum incompatible for generic metrics in these classes, and although nontrivial symmetry does permit the existence of locally causal algebras on the spacelike cylinder, these are not realistic quantum field algebras because their local structure is that of the abelian classical theory.

It remains to prove Lemma 6.1.

Proof of Lemma 6.1: From Eq. (5.13) we have

$$(h_1 \diamond_{\boldsymbol{g}} h_2)(\zeta) = \sum_{\substack{n\in\mathbb{Z}\\ \epsilon=\pm}} i\epsilon e^{in\zeta}\xi_{n\epsilon}(h_1)\xi_{-n-\epsilon}(h_2) \tag{6.8}$$

if $h_1, h_2 \in \mathcal{D}(M^-)$. In particular, if $h_{n\epsilon}\in\mathcal{D}(M^-)$ is chosen so that $\Delta_{\boldsymbol{\eta}} h_{n\epsilon} = \xi_{-n-\epsilon}$ then $(h_{n\epsilon}\diamond_{\boldsymbol{g}} f)(\zeta) = e^{in\zeta}\xi_{-n-\epsilon}(f)$ and therefore

$$[\phi(h_{n\epsilon}),\phi(f)] = i\Psi(e_n)\xi_{-n-\epsilon}(f) \tag{6.9}$$

for all $f\in\mathcal{D}(M^-)$ where $e_n\in\mathcal{D}(G)$ is the function $e_n(\zeta)=e^{in\zeta}$.

For any $p\in M^-$, pick $\epsilon>0$ so that $N_{2\epsilon}(p)$ is a neighbourhood of local causality for $\mathcal{A}$. For any f_1 supported in $N_\epsilon(p)$ we may find f_2 with support spacelike separated from that of f_1 in $N_{2\epsilon}(p)$, and therefore obeying $[\phi(f_1),\phi(f_2)]=0$. Applying the Jacobi identity to f_1, f_2 and $h_{n\epsilon}$, we obtain

$$\begin{aligned} 0 &= [\phi(f_1),[\phi(f_2),\phi(h_{n\epsilon})]] + [\phi(f_2),[\phi(h_{n\epsilon}),\phi(f_1)]] + [\phi(h_{n\epsilon}),[\phi(f_1),\phi(f_2)]] \\ &= -i[\phi(f_1),\Psi(e_n)]\xi_{-n-\epsilon}(f_2) + i[\phi(f_2),\Psi(e_n)]\xi_{-n-\epsilon}(f_1). \end{aligned} \tag{6.10}$$

Eliminating $[\phi(f_2),\Psi(e_n)]$ from these two equations ($\epsilon=\pm$) we have

$$[\phi(f_1),\Psi(e_n)]\left(\xi_{-n-}(f_2)\xi_{-n+}(f_1) - \xi_{-n+}(f_2)\xi_{-n-}(f_1)\right) = 0. \tag{6.11}$$

Assuming that at least one of $\xi_{-n\pm}(f_1)$ is nonzero, we deduce that $\phi(f_1)$ commutes with $\Psi(e_n)$ by varying f_2. If $\xi_{-n-}(f_1)=\xi_{-n+}(f_1)=0$, then Eq. (6.10) becomes

$$[\phi(f_1),\Psi(e_n)]\xi_{-n\pm}(f_2) = 0 \tag{6.12}$$

and we obtain the same conclusion by varying f_2 again. Since f_1 and p were arbitrary we conclude (using a partition of unity and property (Q2)) that this is true for any $f_1\in\mathcal{D}(M^-)$ and therefore for any $f_1\in\mathcal{D}(M)$ using properties of the Cauchy problem and the field equation (Q3). Thus each $\Psi(e_n)$ ($n\in\mathbb{Z}$) commutes with all the generators $\phi(f)$ of $\mathcal{A}$ and is therefore a scalar multiple of the identity. A simple Fourier argument, exploiting continuity of $[\phi(f_1),\phi(f_2)]$ in the f_i, now shows that all commutators are scalar multiples of the identity, from which we obtain $\Psi(\cdot)=\psi(\cdot)\mathbb{1}$ for some $\psi\in\mathcal{D}'(G)$, completing the proof. ■

7 Conclusion

In this paper we have given a full and rigorous treatment of F-local and locally causal algebras for massive Klein–Gordon equation on 2-dimensional cylinder spacetimes whose metrics deviate from $\boldsymbol{\eta}$ within a compact region. We have seen that the only F-local algebra admitted by a generic timelike cylinder is in fact the usual field algebra, and that the locally causal algebras form a 1-parameter family related to the usual algebra by rescaling Planck's constant and/or a reversal of time orientation. Generic spacelike cylinders fail to be compatible with both F-locality and local causality. A full discussion of these results, as well as others concerning 4-dimensional spacelike cylinders will appear in [13] (see also [27]); here we will make only a few brief remarks.

Firstly, the content of our results is that the F-local and locally causal theories coincide generically with the conventional globally hyperbolic theory on cylinder spacetimes. Thus any algebraic framework for quantum field theory on the spacelike cylinders must violate F-locality and even local causality: causality violation on a cosmic scale would in principle be detectable by local measurements.

Secondly, although our results give strong reason to believe that no spacelike cylinders other than the Minkowskian spacelike cylinder $(M, \boldsymbol{\eta}, C^{S+}_{\boldsymbol{\eta}})$ are F-quantum compatible with massive Klein–Gordon theory, we have not completely resolved this point. It may be that a more detailed scattering analysis would shed further light on this issue.

The F-quantum compatibility of $(M, \boldsymbol{\eta}, C^{S+}_{\boldsymbol{\eta}})$ and the generic F-quantum incompatibility of more general spacelike cylinders with massive Klein–Gordon theory shows that F-quantum compatibility can be quite a delicate property. Furthermore, neither the massless 2-dimensional case, nor the situation in 4-dimensions is fully resolved [13, 27]. It is clear, for example, that the massless F-quantum compatibility of $(M, \boldsymbol{\eta}, C^{S+}_{\boldsymbol{\eta}})$ is stable under general conformal perturbations of the metric. A more detailed discussion of the current status of F-locality will appear in [13].

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A Proof of Theorem 5.2

Theorem 5.2 *For generic $\boldsymbol{g} \in \mathcal{G}^{(G)}_K$, we have $\sigma_{\boldsymbol{\eta}}(S_{\boldsymbol{g}}\xi_{-n-\epsilon}, \xi_{n'\epsilon'}) \neq 0$ whenever $n = n'$ (if $G = \mathrm{SO}(2)$) or $n \equiv n' \pmod N$ (if $G = \mathbb{Z}_N$).*

Proof: Recall that a subset of a topological space is said to be generic if it contains a countable intersection of open dense sets. Thus it is enough to show that for each choice

of $n, n', \epsilon, \epsilon'$ with $n \equiv n' \pmod N$ (if $G = \mathbb{Z}_N$) or $n = n'$ (if $G = \mathrm{SO}(2)$), the set

$$\mathcal{N} = \{\boldsymbol{g} \in \mathcal{G}_K^{(G)} \mid \sigma_{\boldsymbol{\eta}}(S_{\boldsymbol{g}}\xi_{-n-\epsilon}, \xi_{n'\epsilon'}) \neq 0\} \tag{A.1}$$

is open and dense in $\mathcal{G}_K^{(G)}$. The set $\mathcal{N}$ is open because $\boldsymbol{g} \mapsto \sigma_{\boldsymbol{\eta}}(S_{\boldsymbol{g}}u, v)$ is continuous for fixed $u, v \in \mathcal{F}_{\boldsymbol{\eta}}$ by Eq. (4.15) and the fact that – as mentioned in Section 4.1 – the map $\boldsymbol{g} \mapsto \Delta_{\boldsymbol{g}} f$ is continuous from $\mathcal{G}_K$ to $\mathcal{E}(M)$ for each fixed $f \in \mathcal{D}(M)$. To establish density of $\mathcal{N}$, we fix $\boldsymbol{g} \in \mathcal{G}_K^{(G)}$ and show that it can be approximated from within $\mathcal{N}$. We will use the following Lemma, which is proved at the end of this appendix.

Lemma A.1 *Let $u^+ = \Omega_{\boldsymbol{g}}^+ \xi_{-n-\epsilon}$ and $v^- = \Omega_{\boldsymbol{g}}^- \xi_{n'\epsilon'}$. Then*

$$\operatorname{supp} u^+ \cap \operatorname{supp} v^- \cap K \tag{A.2}$$

has nonempty interior.

Now because $\boldsymbol{g}$ is G-invariant, we have

$$T_{(0,\zeta)} u^+ = e^{in\zeta} u^+ \qquad \text{and} \qquad T_{(0,\zeta)} v^- = e^{-in'\zeta} v^- \tag{A.3}$$

for $\zeta \in G$. Our conditions on n and n' entail that the product $u^+ v^-$ is G-invariant, so Lemma A.1 allows us to pick a G-invariant $V \in C_0^\infty(\operatorname{int} K; \mathbb{R})$ for which

$$\int_M g^{1/2} u^+ V v^- \neq 0, \tag{A.4}$$

and which defines a 1-parameter family $\{\boldsymbol{g}_\lambda \mid \lambda \in \mathbb{R}\}$ of conformal perturbations of $\boldsymbol{g}$ by

$$\boldsymbol{g}_\lambda = \left(1 + \frac{\lambda V}{\mu}\right) \boldsymbol{g}. \tag{A.5}$$

One may check that $\boldsymbol{g}_\lambda \in \mathcal{G}_K^{(G)}$ for all sufficiently small λ, and $\boldsymbol{g}_\lambda \to \boldsymbol{g}$ as $\lambda \to 0$. Since

$$P_{\boldsymbol{g}_\lambda} = \left(1 + \frac{\lambda V}{\mu}\right)^{-1} (P_{\boldsymbol{g}} + \lambda V) \tag{A.6}$$

the corresponding scattering operators $S_{\boldsymbol{g}_\lambda}$ are equal to those for $P_{\boldsymbol{g}} + \lambda V$ relative to $P_{\boldsymbol{g}}$ for sufficiently small λ. Applying a Born expansion, we deduce that the function $\lambda \mapsto \sigma_{\boldsymbol{\eta}}(S_{\boldsymbol{g}_\lambda}\xi_{-n-\epsilon}, \xi_{n'\epsilon'})$ is differentiable at $\lambda = 0$ with derivative

$$\left.\frac{d}{d\lambda}\sigma_{\boldsymbol{\eta}}(S_{\boldsymbol{g}_\lambda}\xi_{-n-\epsilon}, \xi_{n'\epsilon'})\right|_{\lambda=0} = -\int_M g^{1/2} u^+ V v^-, \tag{A.7}$$

which is nonzero by Eq. (A.4). We infer that $\sigma_{\boldsymbol{\eta}}(S_{\boldsymbol{g}_\lambda}\xi_{-n-\epsilon}, \xi_{n'\epsilon'}) \neq 0$ and therefore $\boldsymbol{g}_\lambda \in \mathcal{N}$ for all sufficiently small $\lambda \neq 0$. Consequently, every neighbourhood of $\boldsymbol{g}$ in $\mathcal{G}_K$ has nontrivial intersection with $\mathcal{N}$, and we conclude that $\mathcal{N}$ is dense as required. ■

Proof of Lemma A.1: We will use the following facts:

1. Since $v^- = \Omega^-_{\boldsymbol{g}} \xi_{n'\epsilon'}$ agrees with $\xi_{n'\epsilon'}$ in $M^+ = M \backslash J^-_{\boldsymbol{\eta}}(K)$ and is therefore bounded away from zero there, v^- is nonvanishing in some neighbourhood of any point on $\partial J^-_{\boldsymbol{\eta}}(K)$ by continuity. Similarly, u^+ is nonvanishing in some neighbourhood of any point on $\partial J^+_{\boldsymbol{\eta}}(K)$.

2. v^- cannot vanish identically in M^- (otherwise it would vanish everywhere in M). As $M^- = M \backslash J^+_{\boldsymbol{\eta}}(K)$ belongs to the past domain of dependence of $\partial J^+_{\boldsymbol{\eta}}(K)$, v^- is nonvanishing in some neighbourhood of a point on $\partial J^+_{\boldsymbol{\eta}}(K)$. Similarly, u^+ is nonvanishing in some neighbourhood of a point on $\partial J^-_{\boldsymbol{\eta}}(K)$.

There are now two cases, depending on the geometry of $K = J^+_{\boldsymbol{\eta}}(\{p\}) \cap J^-_{\boldsymbol{\eta}}(\{p'\})$. Choose points (t_p, z_p), $(t_{p'}, z_{p'})$ in the covering space of M such that $p = q(t_p, z_p)$, $p' = q(t_{p'}, z_{p'})$ and $|z_{p'} - z_p| \leq \pi$. As K is nonempty and $p \neq p'$, we have $t_{p'} > t_p$. Then we have

$$\begin{aligned} J^+_{\boldsymbol{\eta}}(\{p\}) &= \{q(t,z) \mid t - t_p \geq |z - z_p|\} \\ \partial J^+_{\boldsymbol{\eta}}(\{p\}) &= \{q(t_p + \chi, z_p \pm \chi) \mid 0 \leq \chi \leq \pi\} \end{aligned} \tag{A.8}$$

and

$$\begin{aligned} J^-_{\boldsymbol{\eta}}(\{p'\}) &= \{q(t,z) \mid t - t_{p'} \geq |z - z_{p'}|\} \\ \partial J^-_{\boldsymbol{\eta}}(\{p'\}) &= \{q(t_{p'} - \chi, z_{p'} \pm \chi) \mid 0 \leq \chi \leq \pi\}. \end{aligned} \tag{A.9}$$

Case (i): If $t_{p'} - t_p + |z_{p'} - z_p| > 2\pi$ then $\partial J^+_{\boldsymbol{\eta}}(K)$ and $\partial J^-_{\boldsymbol{\eta}}(K)$ do not intersect and $\partial K = \partial J^+_{\boldsymbol{\eta}}(K) \cup \partial J^-_{\boldsymbol{\eta}}(K)$. Using the second fact above, we pick $p'' \in \partial J^+_{\boldsymbol{\eta}}(K)$ such that v^- is nonvanishing in a neighbourhood of p''; by the first fact, u^+ must also be nonvanishing in a neighbourhood of p''. Since $p'' \in \partial K$, the intersection of these neighbourhoods intersects K, and $\operatorname{supp} u^+ \cap \operatorname{supp} v^- \cap K$ has nonempty interior.

Case (ii): If $t_{p'} - t_p + |z_{p'} - z_p| \leq 2\pi$ then $\partial J^+_{\boldsymbol{\eta}}(K) \cap \partial J^-_{\boldsymbol{\eta}}(K) \cap \partial K$ consists of the two points

$$q\left(\frac{t_{p'} + t_p \pm (z_{p'} - z_p)}{2}, \frac{z_{p'} + z_p \pm (t_{p'} - t_p)}{2}\right) \tag{A.10}$$

each of which has a neighbourhood in which both u^+ and v^- are nonvanishing by the first fact above. Hence $\operatorname{supp} u^+ \cap \operatorname{supp} v^- \cap K$ has nonempty interior as required. ∎

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